Two routes to automata minimization and the ways to reach it efficiently

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Common knowledge in FA Theory

- Every regular language L has a *minimal DFA* (that is canonically associated with L)

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$\mathcal{A}$ DFA

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- The minimal quotient of a DFA $\mathcal{A}$ may be effectively computed by the 'Hopcroft' algorithm with a complexity $\mathrm{O}(n \log n)$

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- Under some hypotheses, the Backward algorithm may be improved into the '*Fast Backward*' algorithm with a complexity $\mathrm{O}(m \log n)$

What is this talk about

$\mathcal{A}$ WFA n states m transitions

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- This quotient is sometimes called
  the *bisimulation minimal model* of $\mathcal{A}$

- The minimal quotient of a WFA $\mathcal{A}$ may be effectively computed by the '*Forward*' algorithm with a complexity $\mathrm{O}(m\,n)$

- The minimal quotient of a WFA $\mathcal{A}$ may be effectively computed by the '*Backward*' algorithm with a complexity $\mathrm{O}(m\,n)$

- Under some hypotheses, the Backward algorithm may be improved into the '*Fast Backward*' algorithm with a complexity $\mathrm{O}(m \log n)$

Examples of automata minimisation

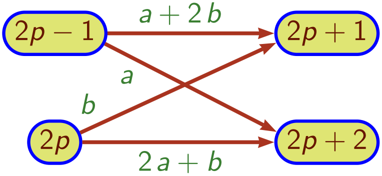
with AWALI

Benchmarks

	k	14	17	20	23	26	30
	F_k	987	4 181	17 711	75 025	317 811	2 178 309
Forward	t (s)	0.42	7.37	139	–		
	$10^{-7} t/F_k^2$	4.3	4.2	4.4			
Backward	t (s)	0.010	0.045	0.257	1.36	73	257
	$10^{-7} t/k\,F_k$	7.2	6.3	7.3	7.6	6.7	7.5
Fast	t (s)	0.006	0.025	0.140	0.70	41	139
Backward	$10^{-7} t/k\,F_k$	4.2	3.5	3.9	3.8	3.5	3.7

Minimisation of $\mathcal{F}_k$

Benchmarks



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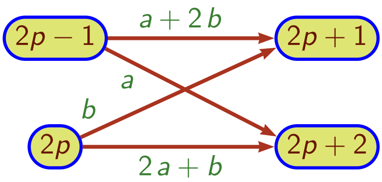

	n	2^{10}	2^{12}	2^{13}	2^{14}	2^{15}	2^{22}
Forward	t (s)	3.29	53.2	214	–		
	$10^{-6}t/n^2$	3.1	3.2	3.2			
Backward	t (s)	0.31	4.92	20.5	86.1	346	–
	$10^{-7}t/n^2$	3.0	2.9	3.1	3.2	3.2	
Fast	t (s)	0.008	0.030	0.061	0.12	0.24	30.8
Backward	$10^{-6}t/n$	7.8	7.3	7.4	7.3	7.3	7.3

Minimisation of Railroad(n)

The theory behind minimisation algorithms

- Automata (DFA, NFA, WFA) are (mathematical) *structures*

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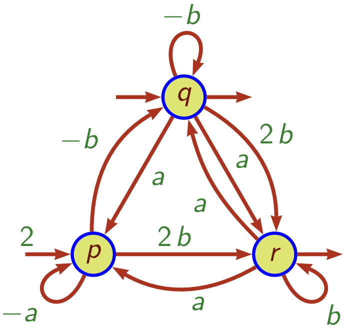
- Automata (DFA, NFA, WFA) are (mathematical) *structures*

- Structures admit *morpisms* $\varphi \colon \mathcal{A} \to \mathcal{B}$, that is,
  maps that respect the structure

- The *kernel* of $\varphi \colon \mathcal{A} \to \mathcal{B}$, that is, the equivalence map of φ is,
  a *partition* of the elements of the structure,
  here the states, that is called a *congruence*

A useful trick

$$\mathcal{A} = \langle\, I, E, T \,\rangle$$



$$\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -a & -b & 2\,b \\ a & -b & a+2\,b \\ a & a & b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

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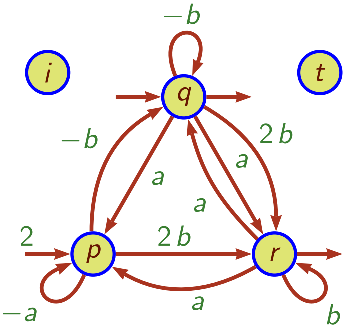

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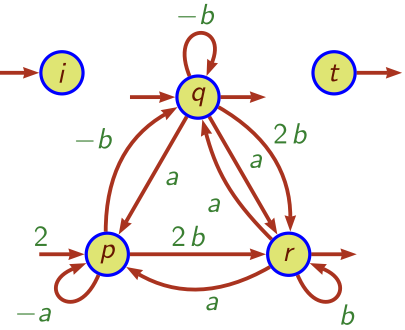

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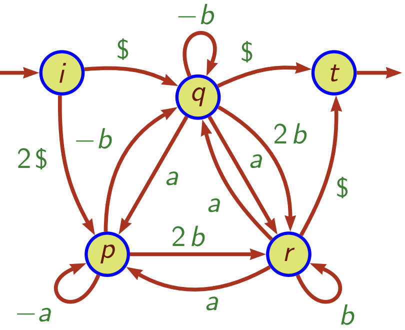

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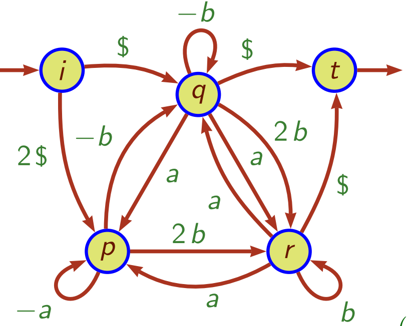
A useful trick

$\mathcal{A} = \langle\, I, E, T \,\rangle \qquad A_{\$} = A \cup \{\$\} \qquad \mathcal{A}_{\$} = \langle\, i, E_{\$}, t \,\rangle$



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Definition

$\mathcal{A} = \langle\, Q, i, E, t \,\rangle$ $\mathbb{K}$-automaton

An equivalence $\mathcal{P}$ on Q is a *congruence* on $\mathcal{A}$, if:

$$\{i\} \in \mathcal{P}\,, \qquad \{t\} \in \mathcal{P}\,, \qquad \text{and}$$

$$\forall p, q \quad p\mathcal{P}q \quad \Longrightarrow \quad \forall a \in A_{\$}\,,\ \forall D \in \mathcal{P} \quad \sum_{r \in D} E(p,a,r) = \sum_{r \in D} E(q,a,r)$$

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The quotient of $\mathcal{A}$ by its *coarsest* congruence is *the minimal quotient* of $\mathcal{A}$

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Remark

The definition of a congruence (and of Out-morphism) is directed

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The definition of a congruence (and of Out-morphism) is directed

The definition of *Out-morphism* coincides

- for DFA, with the classical notion of *morphism*
- for NFA, with the notion of *bisimulation*
- for WFA, with the *simulation* of Bloom-Ésik

The proto-algorithm

Definition

The *signature* of state p of $\mathcal{A}_{\$} = \langle\, Q, i, E, t \,\rangle$ with respect to $D \subseteq Q$ is the *map* $sig[p, D] \colon A_{\$} \to \mathbb{K}$ defined by:

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$split[C, D]$ map equivalence on C of the signature w.r.t. D

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The proto-algorithm

$\mathcal{P} := \mathcal{P}_0$

while there exists a splitting pair (C, D) in $\mathcal{P}$

$\mathcal{P} := \mathcal{P} \wedge split[C, D]$

The Forward Algorithm

$\mathcal{A}_\$ = \langle\, Q, i, E, t \,\rangle$ n states m transitions

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- At round $i+1$, for every $C \in queue$,
  - compute $split[C, D]$ for every $D \in \mathcal{P}_i$
  - put the pieces in $queue$, even if C is not split (but the singletons)

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- $\mathcal{P}_{i+1}$ = content of queue + singletons

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- $\mathcal{P}_{i+1}$ = content of *queue* + singletons
- If no split occurs in round $i+1$, ie if $\mathcal{P}_{i+1} = \mathcal{P}_i$, stop

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Theorem

Forward Algorithm computes the coarsest congruence in $\mathrm{O}(n(m+n))$

The Backward Algorithm

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The Backward Algorithm queue = queue of classes

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The Backward Algorithm

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- $\mathcal{P}_0 = \{i\}, Q, \{t\}$ $\{t\} \to queue$ $Q \to queue$
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  - $\mathcal{P}_{i+1} = \mathcal{P}_i \wedge \cup\, split[C, D]$

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Theorem
Backward Algorithm computes the coarsest congruence in $\mathrm{O}(n(m+n))$

The Fast Backward Algorithm

Hopcroft's algorithm is an improvement of Backward Algorithm for complete DFA

It implements indeed the strategy 'all but the largest' described by Tarjan and Paige

The Fast Backward Algorithm

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Signatures are equipped with a pointwise addition

$$D \cap D' = \emptyset \quad \Longrightarrow \quad sig[p, D \cup D'] = sig[p, D] + sig[p, D']$$

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Definition

$\mathcal{A}$ has *simplifiable signatures* if $\forall D \subseteq Q \quad \forall C \subseteq D \quad \forall p, q \in Q$

$sig[p, D] = sig[q, D]$ and $sig[p, C] = sig[q, C] \Longrightarrow sig[p, D \backslash C] = sig[q, D \backslash C]$.

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If $(\mathbb{K}, +)$ is a *cancellative monoid* (in particular if $\mathbb{K}$ is a ring), then all $\mathbb{K}$-automata have simplifiable signatures.

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The Fast Backward Algorithm $\quad$ *queue* $=$ queue of classes

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  - for every $C \in \mathcal{P}_i$ that is not a singleton,
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    - compute $\mathit{split}[C, D]$
    - if it is a true split, put pieces in *queue* (even singletons)
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Theorem

If $\mathcal{A}$ has simplifiable signatures, then Fast Backward Algorithm computes the coarsest congruence in $\mathrm{O}((m+n)\log n)$

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