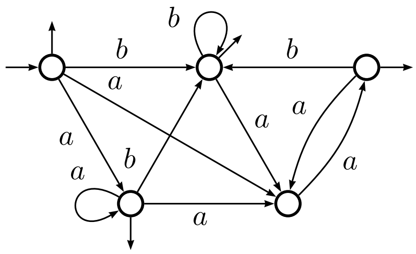
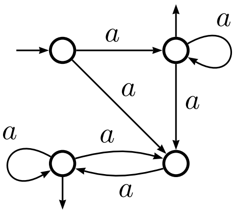
Lecture II — Exercises

Unless stated otherwise, the alphabet A is $A = \{a, b\}$.

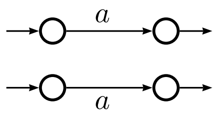
1. Compute the (minimal) quotient of the following $\mathbb{B}$-automaton:



2. Let $\mathcal{D}_1$ be the $\mathbb{B}$-automaton below. Compute the (minimal) quotient of $\mathcal{D}_1$, the co-quotient of $\mathcal{D}_1$, the co-quotient of the quotient of $\mathcal{D}_1$, *etc.*



3. Calculate all the quotients and all the co-quotients of the $\mathbb{N}$-automaton:

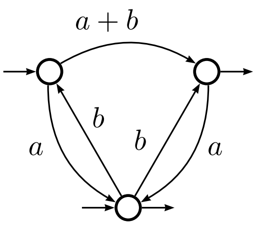


4. **Coloured transition Lemma.** Show the following statement:

Let $\mathcal{A}$ be a (Boolean) automaton on a monoid M the transitions of which are coloured in red or in blue. Then, the set of labels of computations of $\mathcal{A}$ that contain at least one red transition is a rational set (of M).

5. Show that any $\mathbb{Z}$-rational series is the difference of two $\mathbb{N}$-rational series.

6. Construct the Schützenberger covering $\mathcal{S}$ of the following $\mathbb{B}$-automaton $\mathcal{A}$.



How many S-immersions are there in this covering (that is, how many sub-automata $\mathcal{T}$ of $\mathcal{S}$ that are unambiguous and equivalent to $\mathcal{A}$)?

7. Compute the Schützenberger covering of the $\mathbb{B}$-automaton $\mathcal{B}_1$ of the Figure 6.

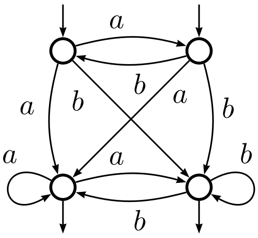


Figure 1: The automaton $\mathcal{B}_1$

8. **Quotients and product of automata.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $\mathbb{K}$-automata on A^*. Show that if $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $\mathcal{B} \odot \mathcal{C}$ is a quotient of $\mathcal{A} \odot \mathcal{C}$.

9. **Quotients and co-quotients of the $\mathcal{C}_n$.**

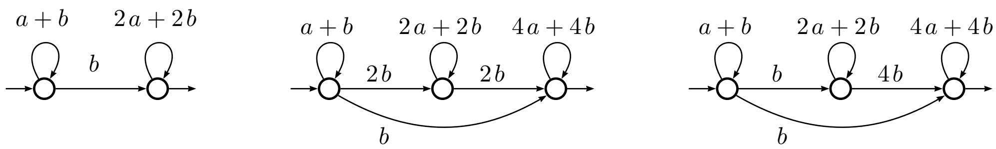
Le $\mathbb{N}$-automate $\mathcal{C}_1$ sur $\{a,b\}^*$ de la Figure 7 (a) associe à chaque mot w l'entier $\overline{w}$ dont la représentation en base 2 est w quand on remplace a par le chiffre 0 et b par 1.

Le $\mathbb{N}$-automate $\mathcal{C}_2$, carré de Hadamard de $\mathcal{C}_1$: $\mathcal{C}_2 = \mathcal{C}_1 \odot \mathcal{C}_1$, a pour quotient minimal $\mathcal{V}_2$ représenté à la Figure 7 (b) et pour co-quotient minimal $\mathcal{V}_2'$ représenté à la Figure 7 (c).

(a) Calculer le quotient minimal $\mathcal{V}_3$ et le co-quotient minimal $\mathcal{V}_3'$ de $\mathcal{C}_3 = \mathcal{C}_2 \odot \mathcal{C}_1$.

(b) Calculer le co-quotient minimal $\mathcal{V}_4'$ de $\mathcal{C}_4 = \mathcal{C}_3 \odot \mathcal{C}_1$. Comparer avec $\mathcal{V}_3'$.

(c) En vous inspirant du calcul précédent, et en vous appuyant sur le calcul du comportement de $\mathcal{C}_{n+1} = \mathcal{C}_n \odot \mathcal{C}_1$, calculer le co-quotient minimal $\mathcal{V}_{n+1}'$ de $\mathcal{C}_{n+1}$ pour tout n.



(a) $\mathcal{C}_1$ (b) $\mathcal{V}_2$ (c) $\mathcal{V}_2'$

Figure 2: Trois $\mathbb{N}$-automates

10. (a) Soient $\mathcal{A}_1$ l'automate (booléen) de la Figure 8 et $\widehat{\mathcal{A}_1}$ son déterminisé. Vérifier que $\widehat{\mathcal{A}_1} \stackrel{X_1}{\Longrightarrow} \mathcal{A}_1$, avec

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

(b) Généralisation. Soient $\mathcal{A}$ un automate (booléen) et $\widehat{\mathcal{A}}$ son déterminisé. Montrer qu'il existe une matrice booléenne X telle que $\widehat{\mathcal{A}} \stackrel{X}{\Longrightarrow} \mathcal{A}$.

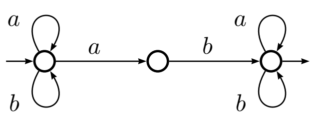


Figure 3: L'automate $\mathcal{A}_1$

11. **Automata with bounded ambiguity and the Schützenberger covering.** In the sequel, $\mathcal{A}$ is a Boolean automaton, $\widehat{\mathcal{A}}$ its determinisation, and $\mathcal{S}$ its Schützenberger covering.

Definition 3. We call *concurrent transition set* of $\mathcal{S}$ a set of transitions which

(i) have the same destination (final extremity),

(ii) are mapped onto the same transition of $\widehat{\mathcal{A}}$.

Two transitions of $\mathcal{S}$ are called *concurrent* if they belong to the same concurrent transition set.

We also set the folllowing definition:

Definition 4. An automaton $\mathcal{A}$ over A^* is of *bounded ambiguity* if there exists an integer k such that every word w in $|\mathcal{A}|$ is the label of at most k distinct computations. The smallest such k is the *ambiguity degree* of $\mathcal{A}$.

(a) What can be said of an automaton whose Schützenberger covering contains no concurrent transitions?

(b) Show that there exists a computation in $\mathcal{S}$ which contains two transitions of the same concurrent transition set if and only if there exists a concurrrent transition which belongs to a circuit.

(c) Let $p \xrightarrow{a} s$ and $q \xrightarrow{a} s$ be two concurrent transitions of $\mathcal{S}$ and

$$c \;:=\; \xrightarrow[\mathcal{S}]{} i \xrightarrow[\mathcal{S}]{x} p \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{y} q \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{z} t \xrightarrow[\mathcal{S}]{}$$

a computation of $\mathcal{S}$ where i is an initial state and t a final state. Show that $w = x\,a\,y\,a\,z$ is the label of at least two computations of $\mathcal{A}$.

(d) Prove that an automaton $\mathcal{A}$ is of bounded ambiguity if and only if no concurrent transition of its Schützenberger covering belongs to a circuit.

(e) Check that $\mathcal{B}_1$ of Figure 6 is of bounded ambiguity.

(f) Give a bound on the ambiguity degree of an automaton as a function of the cardinals of the concurrent transition sets of its Schützenberger covering.

Compute that bound in the case of $\mathcal{B}_1$.

(g) Infer from the above the complexity of an algorithm which decide if an automaton is of bounded ambiguity.