Lecture II

Morphisms of weighted automata

Conjugacy and minimal quotient

In this lecture, we address the problem of finding, given a $\mathbb{K}$-automaton $\mathcal{A}$, a $\mathbb{K}$-automaton $\mathcal{B}$, hopefully of smaller dimension than $\mathcal{A}$, and that inherits the *structure* of $\mathcal{A}$, that is, such that there is a correspondence between the *computations* of $\mathcal{A}$ and those of $\mathcal{B}$. This amounts to describing the *morphisms of* $\mathbb{K}$*-automata*, that is, the mappings between $\mathbb{K}$-automata that *preserve their structure*.

Contents

The classical notion of the minimal automaton of a language, minimal quotient of any determistic that accepts the language is at the same time an example of what we want to generalise and somewhat misleading. Already when it deals with non-deterministic (Boolean) automata, this generalisation requires a *lateralisation* which is not usually associated with the notion morphism and this may may explain it has been given the other name of *bisimulation* in the literature. We call it *Out-morphism* to stress the link with the notion of morphism.

We define Out-morphism in a naive way for non-deterministic Boolean automata and by means of the mathematical notion of *conjugacy* for the general case of weighted automata. This notion being set up, the same theory as the classical one for complete deterministic automata can be rolled out and it is easily seen that every weighted automaton admits a *minimal quotient* as the image of the coarsest Out-morphism which is computed essentially by the same algorithm.

It is worth to be noted that, at least in the case of Boolean automata, the converse operation is indeed at least as interesting: given $\mathcal{A}$, build $\mathcal{B}$ of which $\mathcal{A}$ is a morphic image, hence larger than $\mathcal{A}$, but whose computations are less entangled, in such a way that it becomes possible, by means of other operations, to distinguish and make choices between these computations. Such constructions are essentially considered (even for general weighted automata) when the computations of $\mathcal{B}$ are in a 1-to-1 corespondence with those of $\mathcal{A}$, that is, when $\mathcal{B}$ is a *covering* of $\mathcal{A}$.

1 Morphisms of Boolean automata

This section is more than a reminder or an appetizer. It introduces at the end the notions of *local properties* of morphisms, that will be instrumental in the study of transducers.

In this section, all automata are Boolean automata. We begin with the presentation of the classical definition and computation of the minimal automaton of a rational language while insisting on the morphism point of view.

1.1 The case of (complete) deterministic automata

A deterministic automaton is denoted by $\mathcal{A} = \langle\, A, Q, i, \delta, T \,\rangle$ rather than by $\mathcal{A} = \langle\, A, Q, I, E, T \,\rangle$, where δ is the *transition function*, that is, a map $\delta\colon Q \times A \to Q$. For every w in A^* and p in Q, we write $p \cdot w = q$ rather than $\delta(p, w) = q$. Since $(p \cdot u) \cdot v = p \cdot uv$ the transition function δ defines an *action* of A^* over Q.

The *minimal automaton* of a language L of A^* is defined by means of the *quotient* operation that anticipate the notion of *quotient of a series* (*cf.* Definition III.13): if u is in A^*, the *(left) quotient of* L *by* u is the language $u^{-1}L = \{v \in A^* \mid uv \in L\}$. Let $\mathbf{R}_L$ be the set of quotients of L: $\mathbf{R}_L = \{u^{-1}L \mid u \in A^*\}$; $\mathbf{R}_L$ is finite if and only if L is a rational language.

Since $(uv)^{-1}L = v^{-1}(u^{-1}L)$, the (left) quotient is a (right) action of A^* over the set of languages $\mathfrak{P}(A^*)$, which in turn defines a deterministic automaton on any set of languages closed by quotient.

For every rational language L, let us denote by $\mathcal{A}_L$ the finite deterministic automaton $\mathcal{A}_L = \langle\, A, \mathbf{R}_L, \{L\}, \triangleright, T_L \,\rangle$, where $\triangleright$ is another notation for the quotient:

$$L \triangleright u = u^{-1}L \qquad \text{and} \qquad T_L = \left\{u^{-1}L \;\middle|\; 1_{A^*} \in u^{-1}L\right\} .$$

The automaton $\mathcal{A}_L$ accepts L and is called the *minimal automaton* of L, a terminology that is justified by the following.

Let $\mathcal{A} = \langle A, Q, i, \delta, T \rangle$ be a complete deterministic accessible automaton and $L = L(\mathcal{A})$ the language that it accepts. For all p in Q, we write L_p for the language accepted by the automaton obtained from $\mathcal{A}$ by replacing the initial state i by p:

$$L_p = L(\langle A, Q, p, \delta, T \rangle) = \{w \in A^* \mid p \cdot w \in T\} \quad .$$

Definition 1. The *Nerode equivalence* is the relation ν defined on Q by

$$p \equiv q \mod \nu \iff L_p = L_q \quad .$$

Proposition 2. *The Nerode equivalence induces a map* $\varphi \colon Q \to Q/\nu$ *which saturates* T *and such that* $\varphi(p \cdot a) = (\varphi(p)) \cdot a$ *.*

Proposition 2 allows to define a quotient automaton $\mathcal{A}/\nu = \langle A, Q/\nu, [i]_\nu, \delta_\nu, T_\nu \rangle$.

Theorem 3. $\mathcal{A}_L = \mathcal{A}/\nu$ *.*

Theorem 3 tells at the same time that $\mathcal{A}_L$ is the quotient of *every* complete deterministic automaton that accepts L and that it is the complete deterministic automaton that accepts L with the minimal number of states.

Example 4. Figure 1 shows a complete deterministic automaton and its minimal quotient, obtained by merging states.

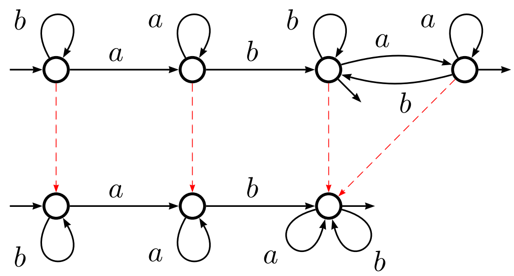


Figure 1: A complete deterministic automaton and its minimal quotient

Proposition 5. *The Nerode equivalence of a finite deterministic automaton is effectively computable by a partition refinement algorithm.*

1.2 The case of general (Boolean) automata

The first definition of morphism for (Boolean) automata follows naturally from the one for deterministic ones. It appears however that it has to be strengthened in order to give rise to the notion of minimal quotient. This new definition of *Out-morphism*, similar to the one of *simulation* for transition systems, applies to any Boolean automaton but is *lateralised* (or *directed*). It is described more systematically in the next subsection.

For the rest of this section, the alphabet is A and fixed, and $\mathcal{A} = \langle Q, I, E, T \rangle$ and $\mathcal{B} = \langle R, J, F, U \rangle$ are two Boolean automata.

Definition 6. A map $\varphi\colon Q \to R$ is a *morphism (of automata)* if:

(i) $\varphi(I) \subseteq J$,
(ii) $\varphi(T) \subseteq U$, and
(iii) for every transition (p, a, q) in E, $(\varphi(p), a, \varphi(q))$ is a transition in F.

If φ is such a morphism, we write $\varphi\colon \mathcal{A} \to \mathcal{B}$.

Proposition 7. *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a morphism, then $|\mathcal{A}| \subseteq |\mathcal{B}|$.*

Example 8. (i) If $\mathcal{U}$ is the one-state automaton which accepts the whole A^*, then the map which sends all states of any automaton $\mathcal{A}$ on the unique state of $\mathcal{U}$ is a morphism.

(ii) If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then both projections $\pi_{\mathcal{A}}\colon \mathcal{C} \to \mathcal{A}$ and $\pi_{\mathcal{B}}\colon \mathcal{C} \to \mathcal{B}$ are morphisms.

The reason for the inclusion in Proposition 7 be strict is that not every (successful) computation in $\mathcal{B}$ may be *lifted* into a (successful) computation in $\mathcal{A}$: the morphism φ is said not to be *conformal*. The two sorts of morphisms in Example 8 are not conformal. Figure 2 gives another example of a non-conformal morphism. It shows that the inclusion in Proposition 7 may be strict, even when φ induces a *bijection* between the transitions — which is the strongest possible condition besides being the identity.

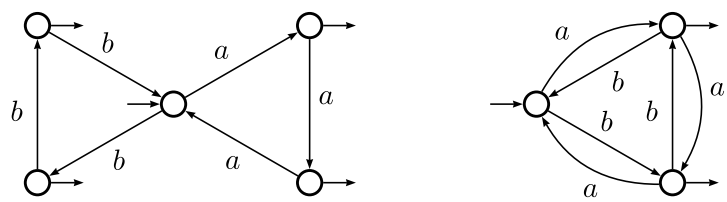


Figure 2: A non-conformal morphism (the morphism is the horizontal projection)

Example 8 (i) shows how weak the notion of automaton morphism can be. In order to have morphisms which really preserve the *structure* of automata (which is supposed to be the role of morphisms) we consider morphisms which meet additional conditions. We first do it 'directly'; in the next subsection, we introduce the more general notion of *local properties* of morphisms that allows to define a richer variety of morphisms.

Definition 9. A map $\varphi\colon Q \to R$ is an *Out-morphism* if:

(o) $\varphi(Q) = R$, that is, if φ is *surjective*,
(i) $\varphi(I) = J$,
(ii) $T = \varphi^{-1}(U)$,
(iii) for every transition (p, a, q) in E, $(\varphi(p), a, \varphi(q))$ is a transition in F,
(iv) for every transition (r, a, s) in F and every p in $\varphi^{-1}(r)$, there exists a q in $\varphi^{-1}(s)$ such that (p, a, q) is a transition in E.

Remark 10. The notion of Out-morphism is *directed* since condition (iv), which consists in a succession of two quantifiers: 'for all..., there exists...', breaks the symmetry between the origin and the destination of the transitions.

Examples 11. (i) If $\mathcal{B}$ is complete and with every state being final, then $\pi_{\mathcal{A}}\colon \mathcal{A}\times\mathcal{B}\to\mathcal{A}$ is an Out-morphism.

(ii) A morphism from a complete deterministic automaton onto an accessible deterministic automaton is an Out-morphism.

Definition 12.
An automaton $\mathcal{B}$ is *a quotient* of $\mathcal{A}$ if there exists an Out-morphism $\varphi\colon \mathcal{A}\to\mathcal{B}$.

Remark 13. The terminology does not make it so clear, but the notion of quotient is *directed* as it derives from the one of Out-morphism. It means somehow that the *true* morphisms are the Out-morphisms.

Proposition 14. *If $\mathcal{B}$ is a quotient of $\mathcal{A}$, then every (successful) computation $\mathcal{B}$ can be lifted into a (successful) computation $\mathcal{A}$.*

Corollary 15. *If $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $|\mathcal{A}|=|\mathcal{B}|$.*

These two statements show that we have reached our goal with the notion of Out-morphism. In order to avoid repetition, we postpone to after the definition of local properties of morphisms and a new expression of Out-morphisms, the presentation of results attached to the notion of quotient.

1.3 Local properties of morphisms

We now take more precise definitions for characterising morphisms; we first set up a convention that reduces the notion of automaton morphism to that of a labelled graph morphism (and get rid of conditions (i) and (ii) in Definitions 6 and 9).

1.3.1 Subliminal states

With every automaton $\mathcal{A}=\langle\, Q,I,E,T\,\rangle$, we associate, by a sort of normalisation, an automaton $\mathcal{A}_{\mathsf{n}}$ to which we have added two new states — $i_{\mathcal{A}}$, an initial state, and $t_{\mathcal{A}}$, a final state — and some transitions, *labelled with* 1_{A^*}, which go from $i_{\mathcal{A}}$ to each initial state of $\mathcal{A}$ and from each final state of $\mathcal{A}$ to $t_{\mathcal{A}}$:

$$\mathcal{A}_{\mathsf{n}}=\langle\, Q\cup\{i_{\mathcal{A}},t_{\mathcal{A}}\},i_{\mathcal{A}},E_{\mathsf{n}},t_{\mathcal{A}}\,\rangle\ ,$$
$$E_{\mathsf{n}}=E\,\cup\,\{(i_{\mathcal{A}},1_{A^*},i)\mid i\in I\}\,\cup\,\{(t,1_{A^*},t_{\mathcal{A}})\mid t\in T\}\ .$$

These two new states, $i_{\mathcal{A}}$ and $t_{\mathcal{A}}$, are called the (initial and final) *subliminal* states of $\mathcal{A}$. We verify easily that $\mathcal{A}_{\mathsf{n}}$ is equivalent to $\mathcal{A}$. More precisely, there is a bijection between the computations of $\mathcal{A}_{\mathsf{n}}$ and those of $\mathcal{A}$ and, of course, the compuatations that correspond in this bijection have the same label.

Remark 16. Even though we deal here with Boolean automata only, the definition of $\mathcal{A}_\mathsf{n}$ may seem to imply a drastic change in the model of (finite) automata since it allows the empty word to be the label of a transition, transitions that are then called *spontaneous transitions* (or ε-transitions). In full generality, this feature opens the possibility for a word to be the label of an infinite number of computations and raises the (difficult) problem of the *validity* when it comes to weighted automata, a problem which will not be treated in these notes. However, if there is *no circuit* of spontaneous transitions in the automaton, then every word is still the label of a *finite number* of computations, its weight can be computed by the sum in Equation (I.1.1) and the behaviour of the automaton is well-defined. Clearly, the construction of $\mathcal{A}_\mathsf{n}$ fall in this case where no circuit of spontaneous transitions is created.

If φ is a map from $\mathcal{A}$ to $\mathcal{B}$, we extend it to a map φ_n from $\mathcal{A}_\mathsf{n}$ to $\mathcal{B}_\mathsf{n}$ by taking $\varphi_\mathsf{n}(i_\mathcal{A}) = i_\mathcal{B}$ and $\varphi_\mathsf{n}(t_\mathcal{A}) = t_\mathcal{B}$. We then verify, just as easily, that $\varphi\colon \mathcal{A} \rightarrow \mathcal{B}$ is an automaton morphism if and only if $\varphi_\mathsf{n}\colon \mathcal{A}_\mathsf{n} \rightarrow \mathcal{B}_\mathsf{n}$ is a labelled graph morphism.

1.3.2 Outgoing and incoming bouquets

For every state p of $\mathcal{A} = \langle\, Q, I, E, T\,\rangle$, we denote by $\mathsf{Out}_\mathcal{A}(p)$ the set of transitions in $\mathcal{A}$ *outgoing* from p and by $\mathsf{In}_\mathcal{A}(p)$ the set of transitions *arriving* at p:

$$\mathsf{Out}_\mathcal{A}(p) = \{e \in E \mid e = (p, a, q)\} \ , \qquad\qquad \mathsf{In}_\mathcal{A}(p) = \{e \in E \mid e = (q, a, p)\} \ ,$$

and we call these sets the *outgoing bouquet* and the *incoming bouquet at state p* respectively (the automaton $\mathcal{A}$ being understood). These notions are *directed*, of course, and *dual*, that is, $\mathsf{In}_\mathcal{A}(p) = \mathsf{Out}_{{}^\mathsf{t}\!\mathcal{A}}(p)$ for every p in Q (with the slight abuse which consists in considering that $\mathcal{A}$ and ${}^\mathsf{t}\!\mathcal{A}$ have the *same* set of transitions). The purpose of the definition of these bouquets is the description of morphism properties based on the remark that if $\varphi\colon \mathcal{A} \rightarrow \mathcal{B}$ is a morphism, then, for every p in Q, φ maps $\mathsf{Out}_\mathcal{A}(p)$ into $\mathsf{Out}_\mathcal{B}(\varphi(p))$ and $\mathsf{In}_\mathcal{A}(p)$ into $\mathsf{In}_\mathcal{B}(\varphi(p))$.

Definition 17. A morphism $\varphi\colon \mathcal{A} \rightarrow \mathcal{B}$ is *Out-surjective* (resp. Out-injective, Out-bijective*) if, for every state p of* $\mathcal{A}_\mathsf{n}$*, the restriction of* φ *to* $\mathsf{Out}_\mathcal{A}(p)$ *is a surjective (resp. injective, bijective) map into* $\mathsf{Out}_\mathcal{B}(\varphi(p))$*.*

The morphism φ *is* In-surjective *(resp.* In-injective, In-bijective*) if, for every state p of* $\mathcal{A}_\mathsf{n}$*, the restriction of* φ *to* $\mathsf{In}_\mathcal{A}(p)$ *is a surjective (resp. injective, bijective) map into* $\mathsf{In}_\mathcal{B}(\varphi(p))$*.*

The 'Out-properties' and the corresponding 'In-properties' are *dual properties*, that is, if $\varphi\colon \mathcal{A} \rightarrow \mathcal{B}$ is Out-surjective (resp. Out-injective, Out-bijective) then $\varphi\colon {}^\mathsf{t}\!\mathcal{A} \rightarrow {}^\mathsf{t}\mathcal{B}$ is In-surjective (resp. In-injective, In-bijective).

Remark 18. Condition (iv) of Definition 9 is another way to express that the morphism φ is Out-surjective.

Remark 19. The conditions of Definition 17 on the outgoing bouquets of the *subliminal initial states* imply that if $\varphi\colon \mathcal{A} \to \mathcal{B}$ is *Out-surjective*, then $\varphi(I) = J$ (condition (i) of Definition 9). Considering the outgoing bouquets of the terminal states — and the transitions toward *subliminal final states* imply that if φ is *Out-surjective*, then $T = \varphi^{-1}(U)$ (condition (ii) of Definition 9). Similarly, if φ is *Out-injective*, then, for every j in J, there exists at most *one* i in I such that $\varphi(i) = j$.

In a dual way, if $\varphi\colon \mathcal{A} \to \mathcal{B}$ is *In-surjective*, then $\varphi(T) = U$ and $I = \varphi^{-1}(J)$ and if φ is *In-injective*, then for every u in U there exists at most *one* t in T such that $\varphi(t) = u$.

Out-morphisms are *conformal*, as expressed by the following.

Proposition 20. *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an Out-surjective morphism. For every path d in $\mathcal{B}$ whose source s is in the image of φ and for every p such that $\varphi(p) = s$ there exists at least one path c in $\mathcal{A}$ whose source is p and such that $\varphi(c) = d$.*

Corollary 21. *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an Out-surjective morphism, then $|\mathcal{A}| = |\mathcal{B}|$.*

Corollary 22. *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an Out-bijective morphism, then φ is a* bijection *between the* successful computations *of $\mathcal{A}$ and those of $\mathcal{B}$.*

Corollary 23. *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an Out-surjective morphism and if $\mathcal{B}$ is* accessible *then φ is (globally)* surjective.

1.3.3 Out- and In-morphisms revisited

With Corollary 23, we see that Out-surjective morphisms are 'almost always' surjective (condition (o) of Definition 9). For simplification and conciseness, in order to avoid special cases, we take that latter property as an hypothesis and set up the following definitions.

Definition 24. A surjective Out-surjective morphism is called an *Out-morphism*.
A surjective In-surjective morphism is called an *In-morphism*.
A surjective Out-bijective morphism is called a *covering*.[1]
A surjective In-bijective morphism is called a *co-covering*.[2]
A surjective Out-injective morphism is called an *immersion*.
A surjective In-injective morphism is called a *co-immersion*.

Remarks 18 and 19 show that Definition 9 and the definition above coincide (for Out-morphisms). We then repeat Definition 12 and Proposition 14.

Definition 25.
An automaton $\mathcal{B}$ is a *quotient* of $\mathcal{A}$ if there exists an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.
An automaton $\mathcal{B}$ is a *co-quotient* of $\mathcal{A}$ if there exists an In-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.

[1] In French, *revêtement*.
[2] In French, *co-revêtement*.

The automaton $\mathcal{B}$ is *a co-quotient* of $\mathcal{A}$ if ${}^t\mathcal{B}$ is a quotient of ${}^t\mathcal{A}$.

Proposition 26. *If $\mathcal{B}$ is a quotient (resp. a co-quotient) of $\mathcal{A}$, then every (successful) computation $\mathcal{B}$ can be lifted onto a (successful) computation $\mathcal{A}$.*

Remark 27. Proposition 26 implies that if $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $\mathcal{A}$ is a *simulation* of $\mathcal{B}$. The terminology of simulation is very common in several areas close to automata theory but using a different vocabulary (transition systems, coalgebra, *etc.*). Note that the converse statement (if $\mathcal{A}$ is a simulation of $\mathcal{B}$, then $\mathcal{B}$ is a quotient of $\mathcal{A}$) does not hold. See Proposition 31 below.

The notion of quotient allows to extend the one of minimal automata.

Proposition 28. *Every automaton $\mathcal{A}$ has a* minimal quotient $\mathcal{C}$*, which is unique up to an isomorphism, and which is the quotient of any quotient $\mathcal{B}$ of $\mathcal{A}$.*

Remark 29. The minimal quotient of an automaton *is not canonically attached to the accepted language* anymore but depends on the automaton it is computed from.

The dual of Proposition 28 also holds.

Proposition 30. *Every automaton $\mathcal{A}$ has a* minimal co-quotient $\mathcal{D}$*, which is unique up to an isomorphism, and which is the co-quotient of any co-quotient $\mathcal{B}$ of $\mathcal{A}$.*

The minimal quotient or co-quotient of an automaton can be computed by a kind of Moore algorithm that consists in successive refinements of the trivial partition on the set of states. We come back to this question at Section 2.3.

Finally, let us note that the notion of quotient allows to give a clean definition of *bisimulation*.

Proposition 31. *Two automata $\mathcal{A}$ and $\mathcal{B}$ are bisimilar if and only if they have the same (or isomorphic) minimal quotient.*

1.4 The Schützenberger covering

We begin with an elementary statement.

Proposition 32. *Let $\mathcal{A}$ be an accessible automaton, $\mathcal{B}$ a complete deterministic automaton equivalent to $\mathcal{A}$, and $\mathcal{E}$ the accessible part of $\mathcal{B} \times \mathcal{A}$. Then $\pi_{\mathcal{A}}$, the projection of $\mathcal{B} \times \mathcal{A}$ onto $\mathcal{A}$, is a covering from $\mathcal{E}$ to $\mathcal{A}$.* ■

Definition 33. Let $\mathcal{A}$ be an accessible automaton and $\widehat{\mathcal{A}}$ its determinisation. The *Schützenberger covering*, or *S-covering*, of $\mathcal{A}$ is the accessible part $\mathcal{S}$ of $\widehat{\mathcal{A}} \times \mathcal{A}$.

Theorem 34. *Let $\mathcal{A}$ be an accessible automaton and $\mathcal{S}$ its Schützenberger covering. Then $\mathcal{S}$ satisfies:*

(i) $\pi_{\mathcal{A}}$ *is a covering from $\mathcal{S}$ to $\mathcal{A}$;*
(ii) $\pi_{\widehat{\mathcal{A}}}$ *is an In-morphism from $\mathcal{S}$ to $\widehat{\mathcal{A}}$.*

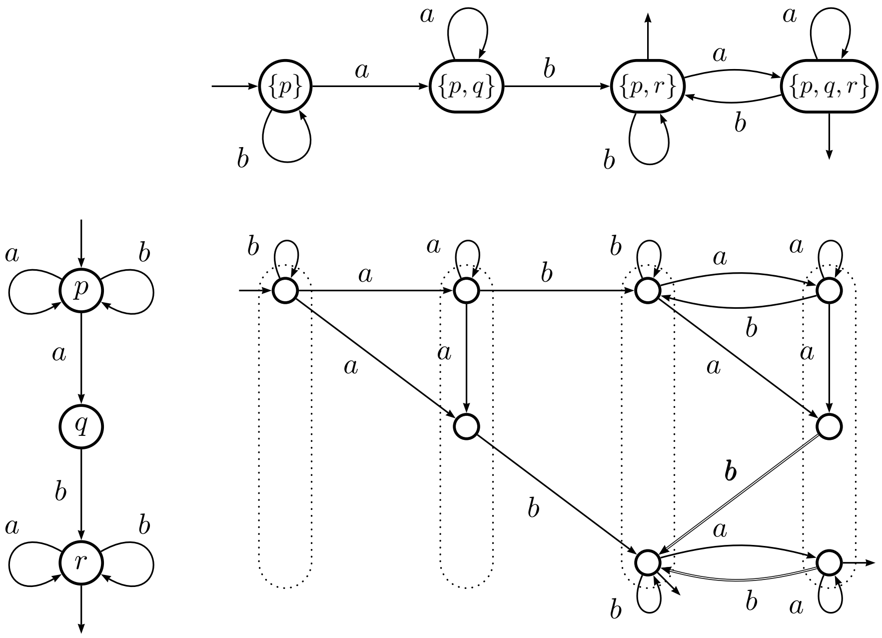


Figure 3: The S-covering of $\mathcal{A}_1$

Example 35. Figure 3 shows the S-covering of the automaton $\mathcal{A}_1$,

Proof of Theorem 34. Since $\widehat{\mathcal{A}}$ is a complete deterministic automaton equivalent to $\mathcal{A}$, condition (i) is the instance of Proposition 32 for $\mathcal{B} = \widehat{\mathcal{A}}$ and it remains to prove condition (ii). From the definition of transitions in $\widehat{\mathcal{A}} = \langle \mathfrak{P}(Q), \{I\}, F, U \rangle$, namely,

$$P \xrightarrow[\widehat{\mathcal{A}}]{a} S \quad \Longleftrightarrow \quad S = \left\{ q \,\middle|\, \exists p \in P \quad p \xrightarrow[\mathcal{A}]{a} q \right\} , \tag{1.1}$$

we first deduce:

Property 36. *The states of $\mathcal{S}$ are the pairs (P, p) where P is a state of $\widehat{\mathcal{A}}$ and p is in P.*

Proof. Let P be a state of $\widehat{\mathcal{A}}$: that is, there exists w in A^* such that

$$P = \left\{ p \,\middle|\, \exists i \in I \quad i \xrightarrow[\mathcal{A}]{w} p \right\} .$$

Thus (P, p) is a state of $\mathcal{S}$; that is, it is accessible in $\widehat{\mathcal{A}} \times \mathcal{A}$ for all p in P. Conversely, if (P, q) is a state of $\mathcal{S}$, there exists w in A^* and i in I such that both $\{I\} \xrightarrow[\widehat{\mathcal{A}}]{w} P$ and $i \xrightarrow[\mathcal{A}]{w} q$, and hence q is in P. ∎

We next deduce by (1.1) that

$$\begin{aligned} \forall P, S \subseteq Q\,,\ \forall q \in S\,,\ \forall a \in A \qquad P \xrightarrow[\widehat{\mathcal{A}}]{a} S \quad &\Longrightarrow \quad \exists p \in P \qquad p \xrightarrow[\mathcal{A}]{a} q \\ &\Longrightarrow \quad \exists p \in P \qquad (P, p) \xrightarrow[\widehat{\mathcal{A}} \times \mathcal{A}]{a} (S, q) \end{aligned}$$

since (P, p) is a state of $\mathcal{S}$, which indeed means that $\pi_{\widehat{\mathcal{A}}} \colon \mathcal{S} \to \widehat{\mathcal{A}}$ is an In-surjective labelled graph morphism.

If $P \subseteq Q$ is final in $\widehat{\mathcal{A}}$ there exists at least one t in P which is final in $\mathcal{A}$, hence a state (P, t) which is final in $\mathcal{S}$. On the other hand, I is the unique initial state of $\widehat{\mathcal{A}}$, every i in I is initial in $\mathcal{A}$, hence every state (I, i) is initial in $\mathcal{S}$. Altogether, $\pi_{\widehat{\mathcal{A}}}$ is an In-surjective morphism. ∎

Corollary 37. *For every Boolean automaton $\mathcal{A}$, there exists an automaton $\mathcal{T}$ such that*

(i) *$\mathcal{T}$ is equivalent to $\mathcal{A}$;*

(ii) *$\mathcal{T}$ is unambiguous;*

(iii) *there exists a morphism $\varphi \colon \mathcal{T} \to \mathcal{A}$.*

It is not a new result that given an automaton $\mathcal{A}$, it is possible to find an unambiguous automaton $\mathcal{T}$ equivalent to $\mathcal{A}$: the determinisation of $\mathcal{A}$ for instance answers the question. That $\mathcal{T}$ can be chosen in a way there is a morphism from $\mathcal{T}$ to $\mathcal{A}$, that is, one can see in $\mathcal{A}$ the computations of $\mathcal{T}$ is a new, and far reaching, property.

2 Morphisms of weighted automata

After the definition of any structure one looks for *morphisms* between objects of that structure, and weighted automata are no exception. Moreover, morphisms of graphs, and therefore of classical Boolean automata, are not less classical, and one waits for their generalisation to weighted automata. Taking into account multiplicity proves however to be not so simple. In the sequel, all automata are supposed to be of finite dimension.[3]

We choose to describe the morphisms of weighted automata via the notion of *conjugacy*, borrowed from the theory of symbolic dynamical systems.

2.1 Conjugacy

Definition 38. A $\mathbb{K}$-automaton $\mathcal{A} = \langle I, E, T \rangle$ *is conjugate to* a $\mathbb{K}$-automaton $\mathcal{B} = \langle J, F, U \rangle$ if there exists a matrix X with entries in $\mathbb{K}$ such that

$$I\,X = J, \qquad E\,X = X\,F, \quad \text{and} \quad T = X\,U.$$

The matrix X is the *transfer matrix* of the conjugacy and we write $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$.

[3] May be it should have been mentioned that the matter developed in Section 1 did not require the automata be finite.

If $\mathcal{A}$ is conjugate to $\mathcal{B}$, then, for every n, the series of equalities holds:

$$I\,E^n\,T = I\,E^n\,X\,U = I\,E^{n-1}\,X\,F\,U = \ldots = I\,X\,F^n\,U = J\,F^n\,U \quad ,$$

from which the following is directly deduced.

Proposition 39. *If $\mathcal{A}$ is conjugate to $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

Example 40. It is easily checked that the $\mathbb{Z}$-automaton $\mathcal{Y}_1$ of Figure 4 is conjugate to the $\mathbb{Z}$-automaton $\mathcal{Z}_1$ of the same figure with the transfer matrix X_1:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

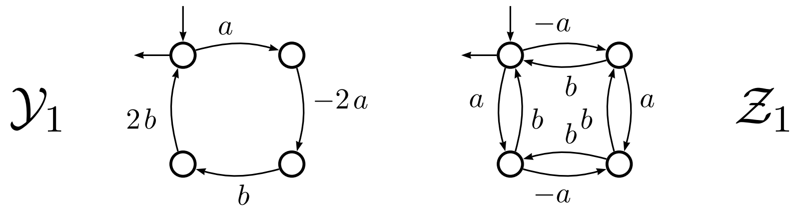


Figure 4: Two conjugate $\mathbb{Z}$-automata

In spite of the idea conveyed by the terminology, the conjugacy relation *is not an equivalence* but a *preorder* relation. Suppose that $\mathcal{A} \xRightarrow{X} \mathcal{C}$ holds; if $\mathcal{C} \xRightarrow{Y} \mathcal{B}$ then $\mathcal{A} \xRightarrow{XY} \mathcal{B}$, but if $\mathcal{B} \xRightarrow{Y} \mathcal{C}$ then $\mathcal{A}$ is not necessarily conjugate to $\mathcal{B}$, and we write $\mathcal{A} \xRightarrow{X} \mathcal{C} \xLeftarrow{Y} \mathcal{B}$ or even $\mathcal{A} \xRightarrow{X}\xLeftarrow{Y} \mathcal{B}$. This being well understood, we shall speak of "conjugate automata" when the orientation does not matter.

If $\mathcal{A} = \langle\, I, E, T \,\rangle$ is conjugate to $\mathcal{B} = \langle\, J, F, U \,\rangle$ then the same conjugacy relation holds between the matrices of the corresponding representations, that is, if $\mathcal{A} = (\, I, \mu, T \,)$ and $\mathcal{B} = (\, J, \kappa, U \,)$, then, as above, $I\,X = J$, $T = X\,U$, and

$$\forall a \in A \qquad \mu(a)\;X = X\,\kappa(a) \quad . \tag{2.1}$$

Then, the same conjugacy relation holds for the representations of every word, that is:

$$\forall w \in A^* \qquad \mu(w)\,X = X\,\kappa(w) \quad . \tag{2.2}$$

2.2 Out-morphisms, In-morphisms

Let $\varphi\colon Q \to R$ be a *surjective* map and X_φ the $Q \times R$-matrix where the (q, r)-th entry is 1 if $\varphi(q) = r$, and 0 otherwise. Since φ is a map, every row of X_φ contains exactly one 1 and since φ is surjective, every column of X_φ contains at least one 1. Such a matrix is called an *amalgamation matrix* in the setting of symbolic dynamics.

Definition 41. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-automata of dimension Q and R respectively. We say that a surjective map $\varphi\colon Q \to R$ is an *Out-morphism* (from $\mathcal{A}$ onto $\mathcal{B}$) if $\mathcal{A}$ is conjugate to $\mathcal{B}$ by X_φ, that is, if $\mathcal{A} \stackrel{X_\varphi}{\Longrightarrow} \mathcal{B}$, and we write $\varphi\colon \mathcal{A} \to \mathcal{B}$.

We also say that $\mathcal{B}$ is a *quotient* of $\mathcal{A}$, if there exists an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.

Remark 42. If $\mathbb{K} = \mathbb{B}$, then Definition 41 coincide with Definition 9.

Again, the notions of Out-morphism and quotient are *lateralised*, or *directed*, since the conjugacy relation is not symmetric. Stated otherwise, and as we see with Proposition 47, it is directed in that it refers not to the transitions of the automaton but to the *outgoing* transitions from the states of the automaton. We then define the *dual* notions of *In-morphism* and *co-quotient*.

Definition 43. With the notation above, a surjective map $\varphi\colon Q \to R$ is an *In-morphism* (from $\mathcal{A}$ onto $\mathcal{B}$) if $\mathcal{B}$ is conjugate to $\mathcal{A}$ by ${}^{\mathrm{t}}X_\varphi$, that is, if $\mathcal{B} \stackrel{{}^{\mathrm{t}}X_\varphi}{\Longrightarrow} \mathcal{A}$, and we write again $\varphi\colon \mathcal{A} \to \mathcal{B}$.

We say that $\mathcal{B}$ is *a co-quotient* of $\mathcal{A}$, if there exists an In-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.

Example 44. Let $\mathcal{C}_2$ be the $\mathbb{N}$-automaton of Figure I.3 and φ_2 the map from $\{j, r, s, u\}$ to $\{i, q, t\}$ such that $j\varphi_2 = i$, $u\varphi_2 = t$ and $r\varphi_2 = s\varphi_2 = q$, then

$$X_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and φ_2 is an Out-morphism from $\mathcal{C}_2$ onto $\mathcal{V}_2$ and an In-morphism from $\mathcal{C}_2$ onto $\mathcal{V}'_2$.

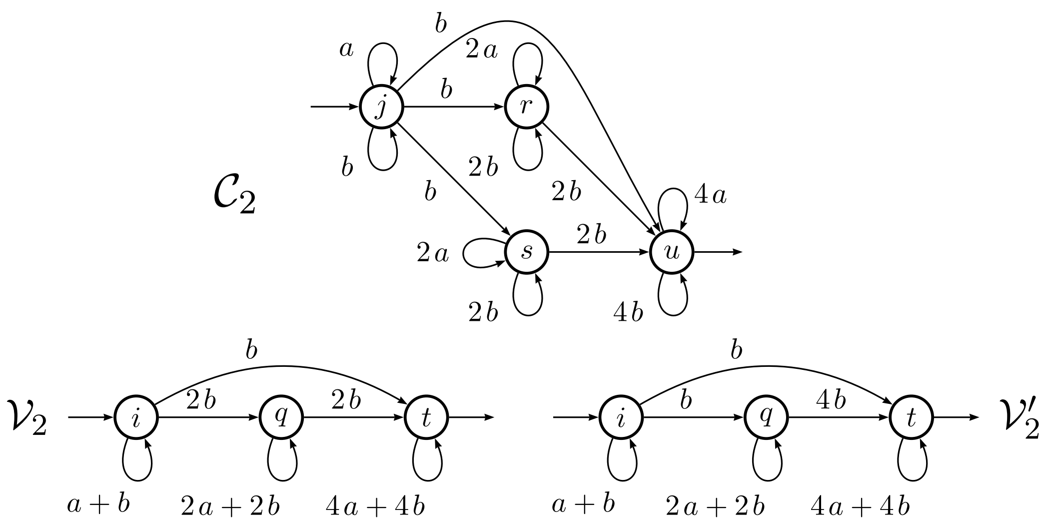


Figure 5: $\mathcal{V}_2$ is a quotient and $\mathcal{V}'_2$ a co-quotient of $\mathcal{C}_2$

In contrast with this special example, a map $\varphi\colon Q \to R$ is not usually both an Out- and an In-morphism. When necessary we shall write $\varphi\colon \mathcal{A} \stackrel{\mathsf{Out}}{\to} \mathcal{B}$ and $\varphi\colon \mathcal{A} \stackrel{\mathsf{In}}{\to} \mathcal{B}$ in order to specify, or to distinguish between, the case.

It directly follows from Definitions 41 and 43 that if $\varphi\colon \mathcal{A} \overset{\mathsf{Out}}{\to} \mathcal{B}$ and $\psi\colon \mathcal{A} \overset{\mathsf{In}}{\to} \mathcal{C}$ are an Out- and an In-morphism respectively, then

$$\mathcal{C} \overset{{}^{\mathrm{t}}X_\psi}{\Longrightarrow} \mathcal{A} \overset{X_\varphi}{\Longrightarrow} \mathcal{B} \qquad \text{hence} \qquad \mathcal{C} \overset{{}^{\mathrm{t}}X_\psi X_\varphi}{\Longrightarrow} \mathcal{B} \ . \tag{2.3}$$

For instance, it holds:

$$\mathcal{V}_2' \overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\Longrightarrow} \mathcal{V}_2 \ .$$

The problem of establishing a converse to the implication expressed in (2.3), that is, proving that if two automata $\mathcal{B}$ and $\mathcal{C}$ are conjugate then there exists an automaton $\mathcal{A}$ such that $\mathcal{B}$ is a quotient of $\mathcal{A}$ and $\mathcal{C}$ a co-quotient of $\mathcal{A}$ is out of the scope of these lecture notes (the answer is indeed somewhat more complex). But we can at lest state the following.

Theorem 45. *Let $\mathbb{K} = \mathbb{B}$ or $\mathbb{N}$. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent $\mathbb{K}$-automata, then there exists a $\mathbb{K}$-automaton $\mathcal{C}$ such that $\mathcal{A}$ is a quotient of $\mathcal{C}$ and $\mathcal{B}$ a co-quotient of $\mathcal{C}$.*

For instance, if $\mathcal{A}$ is a Boolean automaton and $\mathcal{B} = \widehat{\mathcal{A}}$, the Schützenberger covering is the automaton $\mathcal{C}$ the existence of which is insured by the theorem.

Remark 46. The entries of the amalgamation matrix X_φ are $0_\mathbb{K}$ or $1_\mathbb{K}$, hence belong to the center of $\mathbb{K}$ and from (2.1) follows that if φ is an In-morphism from $\mathcal{A}$ to $\mathcal{B}$ it holds

$$\forall a \in A \qquad \kappa(a)\ {}^{\mathrm{t}}X_\varphi = {}^{\mathrm{t}}X_\varphi\,\mu(a) \qquad \text{and then} \qquad {}^{\mathrm{t}}\mu(a)\ X_\varphi = X_\varphi\ {}^{\mathrm{t}}\kappa(a) \ \ ,$$

which means that in the case where we could speak of the transpose of an automaton, hence essentially when $\mathbb{K}$ is commutative, φ is an In-morphism from $\mathcal{A}$ to $\mathcal{B}$ if φ is an Out-morphism from ${}^{\mathrm{t}}\mathcal{A}$ to ${}^{\mathrm{t}}\mathcal{B}$. This statement makes appear more clearly that In-morphism is the dual notion of Out-morphism. Our definition has the advantage that it does not depend on the one of the transpose of an automaton.

It is to be noted that in the definition of an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$, *the image is immaterial* and only counts the map equivalence of φ — which is sufficient to determine the matrix X_φ. From any amalgamation matrix X_φ, we construct a matrix Y_φ by transposing X_φ and by cancelling certain of its entries in such a way that Y_φ is row monomial (with exactly one 1 per row); Y_φ is not uniquely determined by φ but also depends on the choice of a 'representative' in each class for the map equivalence of φ. Whatever this choice, the product $Y_\varphi \cdot X_\varphi$ is the identity matrix of dimension R (as the matrix representing $\varphi \circ \varphi^{-1}$). Easy matrix computations establish the following.

Proposition 47. *Let $\mathcal{A} = \langle\, I, E, T\,\rangle$ be a $\mathbb{K}$-automaton of dimension Q. An equivalence φ on Q is an* Out-morphism *if and only if E and T satisfy the two equations*

$$X_\varphi \cdot Y_\varphi \cdot E \cdot X_\varphi = E \cdot X_\varphi \ \ , \tag{2.4}$$

and

$$X_\varphi \cdot Y_\varphi \cdot T = T \ \ . \tag{2.5}$$

In this case, the $\mathbb{K}$*-automaton* $\mathcal{B} = \langle\, J, F, U\,\rangle$ *defined by the following equations*

$$F = Y_\varphi \cdot E \cdot X_\varphi\,, \qquad J = I \cdot X_\varphi \qquad \textit{and} \qquad U = Y_\varphi \cdot T \tag{2.6}$$

is the quotient of $\mathcal{A}$ *by* φ*.*

Equations 2.4 and 2.5 can be read in the following way: an equivalence φ on Q is an Out-morphism (understood, of $\mathcal{A}$) if for any two states p and p' equivalent modulo φ the *sum* of the labels of the transitions that go from p to *all the states of a whole class* modulo φ is equal to the *sum* of the labels of the transitions that go from p' to the same states *and* if any two entries of T indexed by equivalent states modulo φ are equal, that is (we denote by $[q]_\varphi$ the class of q modulo φ):

$$\forall p, p', q \in Q \qquad p \equiv p' \mod \varphi \quad \Longrightarrow \quad \begin{cases} \text{(i)} & \displaystyle\sum_{r\in[q]_\varphi} E_{p,r} = \sum_{s\in[q]_\varphi} E_{p',s} \\ \text{(ii)} & T_p = T_{p'} \end{cases} \tag{2.7}$$

Remark 48. It is easy to chek that Definitions 12 and 17 coincide if $\mathbb{K} = \mathbb{B}$.

2.3 Minimal quotient

Theorem 49. *Let* $\mathcal{A}$ *be a* $\mathbb{K}$*-automaton of finite dimension. Among all quotients of* $\mathcal{A}$ *(resp. among all co-quotients of* $\mathcal{A}$*), there exists one, unique up to isomorphism and effectively computable from* $\mathcal{A}$*, which has a minimal number of states and which is a quotient (resp. a co-quotient) of all these* $\mathbb{K}$*-automata.*

Proof. A surjective map $\varphi\colon Q \to R$ defines an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ if and only if Equations (2.4) and (2.5) (which do not involve $\mathcal{B}$) are satisfied.

To prove the existence of a minimal quotient, it suffices to show that if $\varphi\colon Q \to R$ and $\psi\colon Q \to P$ are two maps that define Out-morphisms, the map $\omega\colon Q \to S$ also defines an Out-morphism, where $\omega = \varphi \vee \psi$ is the map whose map equivalence is the upper bound of those of φ and ψ; that is, the finest equivalence which is coarser than the map equivalences of φ and ψ. In other words, there exist $\varphi'\colon R \to S$ and $\psi'\colon P \to S$ such that $\omega = \varphi\varphi' = \psi\psi'$ and each class modulo $\omega = \varphi \vee \psi$ can be seen at the same time as a union of classes modulo φ and as a union of classes modulo ψ. It follows that

$$E \cdot X_\omega = E \cdot X_\varphi \cdot X_{\varphi'} = E \cdot X_\psi \cdot X_{\psi'} \tag{2.8}$$

and if two states p and r of Q are congruent modulo ω, there exists q such that $\varphi(p) = \varphi(q)$ and $\psi(q) = \psi(r)$ (in fact a sequence of states q_i *etc.*). The rows p and q of $E \cdot X_\varphi$ are equal, and the rows q and r of $E \cdot X_\psi$ are equal, hence, by (2.8), the rows p and r of $E \cdot X_\omega$ are too.

To compute this minimal quotient we can proceed by successive refinements of partitions, exactly as for the computation of the minimal automaton of a language from a deterministic automaton which recognises the language.

In what follows the maps φ_i are identified with their map equivalences; the image is irrelevant. A state r of Q is identified with the row vector of dimension Q, characteristic of r and treated as such. For example, $\varphi(r) = \varphi(s)$ can be written $r \cdot X_\varphi = s \cdot X_\varphi$.

The map φ_0 has the same map equivalence as T; that is,

$$r \cdot X_{\varphi_0} = s \cdot X_{\varphi_0} \Leftrightarrow r \cdot T = s \cdot T \quad ,$$

which can also be written

$$X_{\varphi_0} \cdot Y_{\varphi_0} \cdot T = T \quad , \tag{2.9}$$

and the same equation holds for every map finer than φ_0. For each i, φ_{i+1} is finer than φ_i and, by definition, r and s are joint in φ_i (that is, $r \cdot X_{\varphi_i} = s \cdot X_{\varphi_i}$) and disjoint in φ_{i+1} if $r \cdot E \cdot X_{\varphi_i} \neq s \cdot E \cdot X_{\varphi_i}$. Let j be the index such that $\varphi_{j+1} = \varphi_j$, that is, such that

$$r \cdot X_{\varphi_j} = s \cdot X_{\varphi_j} \quad \Longrightarrow \quad r \cdot E \cdot X_{\varphi_j} = s \cdot E \cdot X_{\varphi_j} \quad , \tag{2.10}$$

which can be rewritten

$$X_{\varphi_j} \cdot Y_{\varphi_j} \cdot E \cdot X_{\varphi_j} = E \cdot X_{\varphi_j} \quad . \tag{2.11}$$

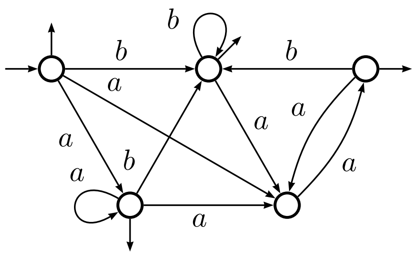
By (2.9) and (2.11), φ_j is an Out-morphism.

Conversely, every Out-morphism ψ satisfies (2.5) and is hence finer than φ_0. Then, for all i, if ψ is finer than φ_i it must also be finer than φ_{i+1}. In fact, if r and s are joint in ψ, it follows that $r \cdot X_\psi = s \cdot X_\psi$ and hence also $r \cdot X_{\varphi_i} = s \cdot X_{\varphi_i}$ since φ_i is coarser than ψ, and hence r and s are joint in φ_{i+1}: ψ is finer than φ_j which is thus the coarsest Out-morphism. ■

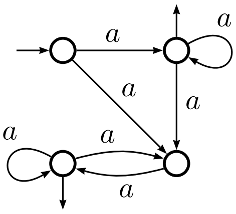
Remark 50. After establishing that the minimal quotient of a $\mathbb{K}$-automaton and the minimal automaton of a language are computed by the *same* algorithm, let us repeat what we already stated in Remark 29: the latter automaton is canonically associated with the language, whereas the former is associated with the $\mathbb{K}$-automaton we started from, and not with its behaviour.

3 Exercises

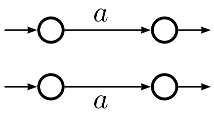
1. Compute the (minimal) quotient of the following $\mathbb{B}$-automaton:



2. Let $\mathcal{D}_1$ be the $\mathbb{B}$-automaton below. Compute the (minimal) quotient of $\mathcal{D}_1$, the co-quotient of $\mathcal{D}_1$, the co-quotient of the quotient of $\mathcal{D}_1$, *etc.*



3. Calculate all the quotients and all the co-quotients of the $\mathbb{N}$-automaton:

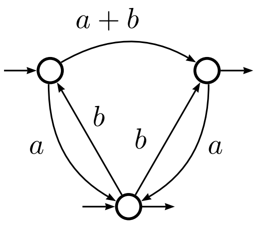


4. **Coloured transition Lemma.** Show the following statement:

Let $\mathcal{A}$ be a (Boolean) automaton on a monoid M the transitions of which are coloured in red or in blue. Then, the set of labels of computations of $\mathcal{A}$ that contain at least one red transition is a rational set (of M).

5. Show that any $\mathbb{Z}$-rational series is the difference of two $\mathbb{N}$-rational series.

6. Construct the Schützenberger covering $\mathcal{S}$ of the following $\mathbb{B}$-automaton $\mathcal{A}$.



How many S-immersions are there in this covering (that is, how many sub-automata $\mathcal{T}$ of $\mathcal{S}$ that are unambiguous and equivalent to $\mathcal{A}$)?

7. Compute the Schützenberger covering of the $\mathbb{B}$-automaton $\mathcal{B}_1$ of the Figure 6.

8. **Quotients and product of automata.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $\mathbb{K}$-automata on A^*. Show that if $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $\mathcal{B} \odot \mathcal{C}$ is a quotient of $\mathcal{A} \odot \mathcal{C}$.

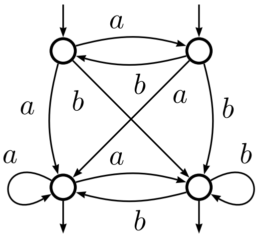


Figure 6: The automaton $\mathcal{B}_1$

9. **Quotients and co-quotients of the $\mathcal{C}_n$.**

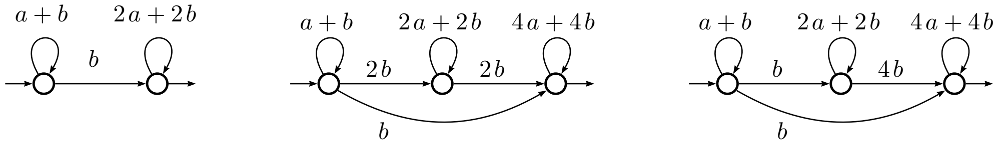
Le $\mathbb{N}$-automate $\mathcal{C}_1$ sur $\{a,b\}^*$ de la Figure 7 (a) associe à chaque mot w l'entier $\overline{w}$ dont la représentation en base 2 est w quand on remplace a par le chiffre 0 et b par 1.

Le $\mathbb{N}$-automate $\mathcal{C}_2$, carré de Hadamard de $\mathcal{C}_1$: $\mathcal{C}_2 = \mathcal{C}_1 \odot \mathcal{C}_1$, a pour quotient minimal $\mathcal{V}_2$ représenté à la Figure 7 (b) et pour co-quotient minimal $\mathcal{V}_2'$ représenté à la Figure 7 (c).

(a) Calculer le quotient minimal $\mathcal{V}_3$ et le co-quotient minimal $\mathcal{V}_3'$ de $\mathcal{C}_3 = \mathcal{C}_2 \odot \mathcal{C}_1$.

(b) Calculer le co-quotient minimal $\mathcal{V}_4'$ de $\mathcal{C}_4 = \mathcal{C}_3 \odot \mathcal{C}_1$. Comparer avec $\mathcal{V}_3'$.

(c) En vous inspirant du calcul précédent, et en vous appuyant sur le calcul du comportement de $\mathcal{C}_{n+1} = \mathcal{C}_n \odot \mathcal{C}_1$, calculer le co-quotient minimal $\mathcal{V}_{n+1}'$ de $\mathcal{C}_{n+1}$ pour tout n.



(a) $\mathcal{C}_1$ (b) $\mathcal{V}_2$ (c) $\mathcal{V}_2'$

Figure 7: Trois $\mathbb{N}$-automates

10. (a) Soient $\mathcal{A}_1$ l'automate (booléen) de la Figure 8 et $\widehat{\mathcal{A}_1}$ son déterminisé. Vérifier que $\widehat{\mathcal{A}_1} \stackrel{X_1}{\Longrightarrow} \mathcal{A}_1$, avec

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

(b) Généralisation. Soient $\mathcal{A}$ un automate (booléen) et $\widehat{\mathcal{A}}$ son déterminisé. Montrer qu'il existe une matrice booléenne X telle que $\widehat{\mathcal{A}} \stackrel{X}{\Longrightarrow} \mathcal{A}$.

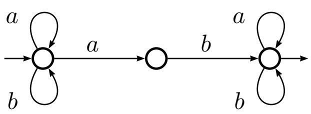


Figure 8: L'automate $\mathcal{A}_1$

11. **Automata with bounded ambiguity and the Schützenberger covering.** In the sequel, $\mathcal{A}$ is a Boolean automaton, $\widehat{\mathcal{A}}$ its determinisation, and $\mathcal{S}$ its Schützenberger covering.

Definition 51. We call *concurrent transition set* of $\mathcal{S}$ a set of transitions which

(i) have the same destination (final extremity),

(ii) are mapped onto the same transition of $\widehat{\mathcal{A}}$.

Two transitions of $\mathcal{S}$ are called *concurrent* if they belong to the same concurrent transition set.

We also set the folllowing definition:

Definition 52. An automaton $\mathcal{A}$ over A^* is of *bounded ambiguity* if there exists an integer k such that every word w in $|\mathcal{A}|$ is the label of at most k distinct computations. The smallest such k is the *ambiguity degree* of $\mathcal{A}$.

(a) What can be said of an automaton whose Schützenberger covering contains no concurrent transitions?

(b) Show that there exists a computation in $\mathcal{S}$ which contains two transitions of the same concurrent transition set if and only if there exists a concurrrent transition which belongs to a circuit.

(c) Let $p \xrightarrow{a} s$ and $q \xrightarrow{a} s$ be two concurrent transitions of $\mathcal{S}$ and

$$c \;:=\; \xrightarrow[\mathcal{S}]{} i \xrightarrow[\mathcal{S}]{x} p \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{y} q \xrightarrow[\mathcal{S}]{a} s \xrightarrow[\mathcal{S}]{z} t \xrightarrow[\mathcal{S}]{}$$

a computation of $\mathcal{S}$ where i is an initial state and t a final state. Show that $w = x\,a\,y\,a\,z$ is the label of at least two computations of $\mathcal{A}$.

(d) Prove that an automaton $\mathcal{A}$ is of bounded ambiguity if and only if no concurrent transition of its Schützenberger covering belongs to a circuit.

(e) Check that $\mathcal{B}_1$ of Figure 6 is of bounded ambiguity.

(f) Give a bound on the ambiguity degree of an automaton as a function of the cardinals of the concurrent transition sets of its Schützenberger covering.

Compute that bound in the case of $\mathcal{B}_1$.

(g) Infer from the above the complexity of an algorithm which decide if an automaton is of bounded ambiguity.

Notation Index

General Index